

hep-th/0005166

TIFR/TH/00-22

May, 2000

D-brane gauge theories from toric singularities of the form \mathbf{C}^3/Γ and \mathbf{C}^4/Γ

Tapobrata Sarkar¹

*Department of Theoretical Physics,
Tata Institute of Fundamental Research,
Homi Bhabha Road, Mumbai 400 005, India*

Abstract

We discuss examples of D-branes probing toric singularities, and the computation of their world-volume gauge theories from the geometric data of the singularities. We consider several such examples of D-branes on partial resolutions of the orbifolds $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ and $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

¹Email: tapo@theory.tifr.res.in

1 Introduction

In the last few years, a lot of progress has been made in the study of D-branes as probes of sub-stringy geometry. While the usual picture of space-time is supposed to be valid upto the string scale, it was argued in [1] that D-branes are the natural candidates to probe the geometry of space-time beyond the string scale. As has been widely accepted by now, the standard concepts of space-time appears, from the D-brane perspective, as the vacuum moduli space of its world volume gauge theory. Indeed, it has been found that the space-time appearing in D-brane probes at ultra-short distance scales has qualitatively different features from that probed by bulk strings [2].

Of particular interest has been the study of D-branes on Calabi-Yau manifolds. In [3],[4], the moduli space of Calabi-Yau manifolds was studied using fundamental strings. A rich underlying geometric structure was discovered, and the moduli space was shown to contain topologically distinct geometric (Calabi-Yau) phases, as well as non-geometric (Landau-Ginzburg) phases. The Kähler sector of the vacuum moduli space was shown to consist of several domains (separated by singular Calabi-Yau spaces), whose union is isomorphic to the complex structure moduli space of the mirror manifold. It was also shown, that using mirror symmetry as in [3] or the approach of gauged linear sigma models as in [4], one can interpolate smoothly between the topologically distinct points in the Kähler moduli space. In the fundamental string picture, the geometric and non-geometric phases of Calabi-Yau manifolds appear as two distinct possibilities, with the latter being thought of as an analytic continuation of the geometric phases in the presence of a non-zero theta angle in the language of [4]. It was, however, shown in [5] that the presence of additional open string sectors change this picture considerably, and that the D-brane linear sigma model probes only a part of the full linear sigma model vacuum moduli space, namely the geometric phase. It was explicitly demonstrated in [5] that D-branes (in the particular examples of the orbifold $\mathbf{C}^3/\mathbf{Z}_3$ and $\mathbf{C}^3/\mathbf{Z}_5$) ‘project out’ the non-geometric phase, in agreement with the results obtained in [6]. This was found to be generally true for Calabi-Yau orbifolds of the form $\mathbf{C}^3/\mathbf{Z}_n$ [7].

Since the pioneering work of [8], which dealt with D-branes on Abelian orbifold singularities of the form $\mathbf{C}^2/\mathbf{Z}_n$ that was generalised to arbitrary non-Abelian singularities of the form \mathbf{C}^2/Γ (Γ being a subgroup of $SU(2)$), D-branes on Abelian and non-Abelian orbifold backgrounds have been ex-

tensively studied in [7],[9], [10],[11],[12],[13]. D-branes at orbifolded conifold singularities have been considered in [14]. (See also [15] for an analysis of a blowup of the four-dimensional $N = 1$, \mathbf{Z}_3 orientifold). Of particular interest has been the applications of methods of toric geometry [16],[17] to the study of D-brane gauge theories on such singularities. In the approach pioneered in [5], the matter content and the interactions of the D-brane gauge theory, which are specified by the D-terms and the F-terms of the gauge theory respectively, are treated on the same footing, and the gauge theory information can be encoded as algebraic equations of the toric variety. It is an interesting question to ask, if this procedure can be reversed, i.e, given a particular toric singularity, is there a way of consistently reading off the world volume gauge theory data for a D-brane that probes this singularity. A step in this direction has been recently taken in [21]. The authors of [21] have given an algorithm by which the geometrical data encoded in a toric diagram can be used to construct the matter content and superpotential interaction of the D-brane gauge theory that probes it, and have demonstrated this in the cases of the suspended pinch point (SPP) singularity of the blowup of the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold and the various partial resolutions of the $\mathbf{C}^3/\mathbf{Z}_3 \times \mathbf{Z}_3$ orbifold analysed in [22]. In the present paper, we critically examine this procedure in several other examples. We first consider this algorithm, which we call the inverse toric procedure, for partial resolutions of the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$. Further, we consider several blowups of the orbifolds $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ and $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and present results on the extraction of the D-brane gauge theory data from the given toric geometric data.

The organisation of the paper is as follows. In section 2, we first recapitulate the essential details of the construction of the D-brane gauge theory on partial resolutions of the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ [19],[20] that will also set the notation and conventions to be used in the rest of the paper and then demonstrate the inverse toric algorithm to construct the gauge theory of a D-brane probing the \mathbf{Z}_2 and the conifold singularities obtained by partially resolving the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold. Section 3 deals with the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ and its various partial resolutions. In section 4, we consider the example of the orbifold $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and consider some resolutions of the same. Section 5 contains discussions and directions for future work.

2 D-branes on $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ and its blowups

We begin this section by recapitulating the essential details of the application of toric methods to the construction of D3-branes on $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ and some of its partial resolutions which will also set the notations and conventions to be followed in the rest of the paper. Recall that the world volume supersymmetric gauge theory of D-branes probing a toric singularity are characterised by its matter content and interactions. While the former are given by the D-term equations, the latter are specified, via the superpotential, as the F-term equations. These two sets of equations, in conjunction, define the moduli space of the theory. In order to define the D-brane moduli space by toric methods, the standard procedure [5] is to re-express the F and D-term constraints in terms of a set of homogeneous variables p_α , and a concatenation of these two sets of equations give rise to the toric data. Let us briefly review this construction for the example of D-3 branes on the orbifold $\mathbf{C}^3/(\mathbf{Z}_2 \times \mathbf{Z}_2)$.

We start by considering the field theory of a set of D-3 branes at a $\mathbf{C}^3/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ singularity. Generically, a single D-p brane of type II string theory at a point in the orbifold \mathbf{C}^3/Γ is constructed by considering the theory of $|\Gamma|$ D-p branes in \mathbf{C}^3 and then projecting to \mathbf{C}^3/Γ where the projection is defined by a combined action of Γ on \mathbf{C}^3 as well as the D-brane Chan-Paton index. The unbroken supersymmetry in $d = 4$ would then be $N = 2$ in the closed string sector while the open string sector will have $N = 1$ supersymmetry. In our example, a generator g of the discrete group $\mathbf{Z}_2 \times \mathbf{Z}_2$ acts simultaneously on the coordinates of \mathbf{C}^3 and also the D-3 brane Chan-Paton factors. The surviving fields in the theory are the ones that are left invariant by a combination of these two actions. Specifically, denoting the action of the quotienting group on \mathbf{C}^3 by $R(g)$ and that on the Chan-Paton indices by $\gamma(g)$ (where γ denotes a regular representation of g), the surviving components of the scalars X that live on the brane world volume are those for which $X = R(g)\gamma(g)X\gamma(g)^{-1}$, and the components of the gauge fields A that survive the projection are those that satisfy $A = \gamma(g)A\gamma(g)^{-1}$. After imposing these projections, the $N = 1$ matter multiplet of the theory consists of the twelve fields

$$(X_{13}, X_{31}, X_{24}, X_{42}), (Y_{12}, Y_{21}, Y_{34}, Y_{43}), (Z_{14}, Z_{41}, Z_{23}, Z_{32}) \quad (1)$$

Where we have denoted the complex bosonic fields corresponding to the

position coordinates tangential to \mathbf{C}^3 by $X^1 = X, X^2 = Y, X^3 = Z$. After projection of the gauge field, the unbroken gauge symmetry in this case is the subgroup $U(1)^4$ of $U(4)$, and, apart from an overall $U(1)$, we have a $U(1)^3$ theory. The total charge matrix is given by

$$d = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \end{pmatrix} \quad (2)$$

With the last row of d signifying the overall $U(1)$. The D-term constraints are now given by

$$\sum_{\mu} d_{(i)\mu}^{\beta} |X_{\beta}^{\mu}|^2 - \zeta_i = 0 \quad (3)$$

where $d_{(i)\mu}^{\beta}$ is the charge of X_{β}^{μ} under the i 'th $U(1)$ and β signifies the matrix indices of the surviving components of X^{μ} . $\sum_i \zeta_i = 0$ is the condition for the existence of supersymmetric vacua.

The superpotential of the theory is given by the expression

$$W = \text{Tr} [X^1, X^2] X^3 \quad (4)$$

With the vacuum satisfying

$$\frac{\partial W}{\partial X^{\mu}} = 0; \quad \text{i.e.} \quad [X^{\mu}, X^{\nu}] = 0 \quad (5)$$

Using the expressions for the X^{μ} 's from eq.(1) in eq. (5), one can derive twelve constraints on the twelve fields X_{ij}, Y_{ij}, Z_{ij} , out of which six are seen to be independent, hence the twelve initial fields can be solved in terms of six independent fields. Denoting the twelve matter fields collectively by $X_i, i = 1 \cdots 12$, and the set of six independent fields (which we take to be $X_{13}, X_{24}, Y_{21}, Y_{34}, Z_{14}, Z_{32}$) by $x_b, (b = 1 \cdots 6)$, the solution, which is of the form

$$X_i = \prod_{b=1}^6 x_b^{K_{ib}} \quad (6)$$

can be encoded in the columns of the matrix K , given by

$$K = \begin{pmatrix} & X_{13} & X_{24} & X_{31} & X_{42} & Y_{12} & Y_{21} & Y_{34} & Y_{43} & Z_{14} & Z_{23} & Z_{32} & Z_{41} \\ X_{13} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ X_{24} & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ Y_{21} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ Y_{34} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ Z_{14} & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ Z_{32} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (7)$$

The columns of the above matrix are vectors in the lattice $N = \mathbf{Z}^6$, and define the edges of a cone σ . In order to make the description of the toric variety explicit, we consider the dual cone σ^\vee , which is defined in the dual lattice $M = \text{Hom}(N, \mathbf{Z})$. The dual cone σ^\vee is defined to be the set of vectors in the dual lattice M which have non-negative inner products with the vectors of σ , i.e

$$\sigma^\vee = \{\mathbf{m} \in M : \langle \mathbf{m}, \mathbf{n} \rangle \geq 0 \ \forall \mathbf{n} \in \sigma\} \quad (8)$$

The dual cone corresponds to defining a new set of fields $p_\alpha (\alpha = 1 \cdots c)$, in terms of which the variables x_b are solved, in a way that is consistent with the relation (6). In this example, $c = 9$ and the dual cone is generated by the columns of the matrix

$$T = \begin{pmatrix} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ X_{13} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{24} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ Y_{21} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ Y_{34} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ Z_{14} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ Z_{32} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad (9)$$

The relationship between p_α and x_b are then of the form

$$x_b = \prod_{\alpha} p_{\alpha}^{T_{b\alpha}} \quad (10)$$

Thus, there are nine variables p_α in terms of which we parametrize the six independent variables x_b . This parametrization, however, has redundancies, and in order to eliminate these, we introduce a new set of \mathbf{C}^* actions on

the p_α 's, with the condition that under these actions, the fields x_b are left invariant. This can be equivalently described by the introduction of a new set of $U(1)$ gauge symmetries, such that the x_b 's are gauge invariant [3]. In our example, since there are nine fields p_α and six independent variables x_b , we introduce a gauge group $U(1)^3$, such that the p_α 's are charged with respect to these, with charges $Q_{n\alpha}$ for the n th $U(1)$. The gauge invariance condition demands that the charge matrix Q must obey

$$TQ^t = 0 \quad (11)$$

where Q^t is the transpose of Q . From the matrix T given above, the charge matrix is given by

$$Q = \begin{pmatrix} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ U(1)_1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ U(1)_2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ U(1)_3 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

We now consider the constraints imposed on the original fields X_i via the D-flatness conditions, and impose these conditions in terms of the new fields p_α . The charges of the fields x_b under the original $U(1)$ charge assignments are given by the matrix

$$V = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad (13)$$

From the relationship $x_b = \prod_\alpha p_\alpha^{T_{b\alpha}}$, it is clear that in order for the charge assignment $q_{i\alpha}^p$ for p_α (corresponding to the original set of $(i = 3)$ $U(1)$ symmetries) to reproduce that for the x_b 's, we must have $q^p T^t = V$, and one possible choice for the matrix q^p is $q_{i\alpha}^p = (VU)_{i\alpha}$ with the matrix U satisfying $TU^t = \text{Id}$. The matrix U is thus the inverse of the matrix T^t . However, since T^t is a rectangular matrix the usual definition of its inverse (as in the case of a square matrix) is not applicable and there are several choices for U . In particular, for a rectangular matrix A , using the rules of single value decomposition, we can compute the Moore-Penrose inverse of A which is defined in a way that the sum of the squares of all the entries

in the matrix $(AA^{-1} - \text{Id})$ is minimised, where Id is the identity matrix in appropriate dimensions. For any choice of the inverse of T^t , the final toric data will of course be identical, and in this section and the next, we simply use the results of [23] in defining the matrix U . For this case, the matrix U is

$$U = (T^t)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Now, multiplying by V and concatenating Q and VU we obtain the total charge matrix,

$$Q_t = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{pmatrix} \quad (14)$$

where the ζ_i are the Fayet-Illiopoulos parameters for the original $U(1)$'s, there being three independent ζ_i 's because of the constraint $\sum_{i=1}^4 \zeta_i = 0$. The toric data can now be calculated from the cokernel of the transpose of Q_t , which (after a few row operations) become

$$\tilde{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (15)$$

Now from the dual cone T , we can, via the matrix K , find expressions for all the twelve initial fields in terms of the fields p_α . These are then used to define invariant variables in the following way

$$\begin{aligned} X_{13}X_{31} = x &= p_1p_2^2p_3p_8p_9 \\ Y_{12}Y_{21} = y &= p_1p_3p_4p_5^2p_6 \\ Z_{14}Z_{41} = z &= p_4p_6p_7^2p_8p_9 \\ X_{13}Y_{34}Z_{41} = w &= p_1p_2p_3p_4p_5p_6p_7p_8p_9 \end{aligned} \quad (16)$$

which follow the relation $xyz = w^2$. The toric diagram of the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ is shown in figure 1(a). Let us now consider some examples of the possible partial resolutions of this space. We will discuss three possible partial blowups as in [23],[25].

The first example corresponds to blowing up the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ by a \mathbf{P}_1 parametrised by $w' = \frac{w}{z}$. This gives the suspended pinch point (SPP) singularity given by $xy = zw'^2$. The SPP can be further blown up by introducing a second \mathbf{P}_1 . One possibility is to parametrise this \mathbf{P}_1 by $y' = \frac{y}{w'}$ (or $x' = \frac{x}{w'}$) and the remaining singularity is the conifold $x'y = zw'$ (or $xy' = zw'$) of figure 1(b). One can also introduce a \mathbf{P}_1 parametrised by say $x' = \frac{x}{z}$ whence the resulting blowup is a $\mathbf{C}^2/\mathbf{Z}_2 \times \mathbf{C}$. Two possible toric diagrams of the latter are shown in fig. 1(c) and fig. 1(d).

Let us now come to the description of the inverse toric problem. Suppose we are given with a singularity depicted by a toric diagram of the form given in fig. 1(b), 1(c) or 1(d). We wish to extract the gauge theory data of a D-brane that probes such a singularity, starting from figure 1(a). This issue was addressed in [21], and let us review the basic steps in their algorithm.

Recall that the toric data for a given singularity that a D-brane probes is given by the matrix \tilde{T} which is the transpose of the kernel of the total charge matrix (obtained by concatenating the F and D-term constraints expressed in terms of the homogeneous coordinates p_α). Hence, the solution to the inverse problem would imply the construction of an appropriately reduced total charge matrix Q_t^{red} , the cokernel of the transpose of which is the reduced matrix \tilde{T}_{red} obtained by deleting the appropriate columns (corresponding to the fields that have been resolved) from \tilde{T} . However, although \tilde{T}_{red} is known, there is no unique way of specifying the reduced charge matrix Q_t^{red} from a knowledge of \tilde{T}_{red} . Further, there is no guarantee apriori that such a Q_t^{red} , if obtained, would continue to describe the world volume theory of a D-brane that probes the given singularity. The algorithm of [21] gives a canonical method by which these issues can be addressed. It consists of determining the fields to be resolved by tuning the Fayet-Illiopoulos parameters appropriately (so that the resulting theory at the end is still a physical D-brane gauge theory) and then reading off the the matrix Q_t^{red} which can be separated into the F and D-terms and thus can be used to obtain the matter content and the superpotential determining the gauge theory. Let us discuss this in some more details. The toric diagrams for the partially resolved singularities are obtained by the deletion of of a subset of

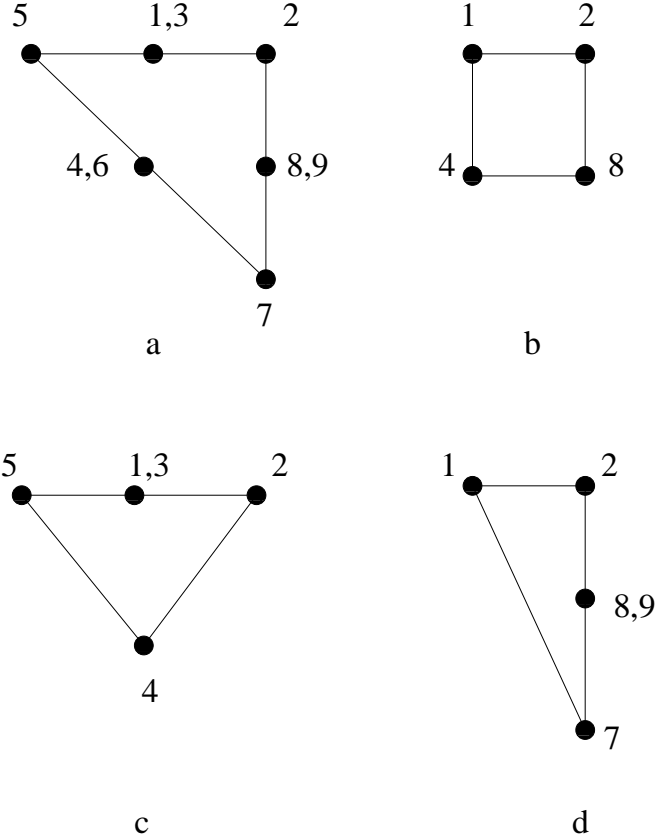


Figure 1: Toric diagram showing a) the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold, b) a resolution to the conifold, c) resolution to the \mathbf{Z}_2 orbifold singularity and d) a second resolution to the \mathbf{Z}_2 orbifold singularity. We have also marked the fields that remain dynamical in these resolutions.

nodes from the original one. Conversely, a given toric diagram of dimension k can be embedded into a singularity of the form $\mathbf{C}^k/\Gamma(k, n)$ where $\Gamma(k, n) = \mathbf{Z}_n \times \mathbf{Z}_n \times \cdots (k-1 \text{ times}) \cdots \mathbf{Z}_n$. In our case, the singularities mentioned above can be embedded into the toric diagram for the singularity $\mathbf{C}^3/\Gamma(3, 2)$. For example, the SPP, and the conifold singularity can both be embedded into the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ singularity, as is obvious from their toric diagrams. We shall henceforth refer to these singularities, in which we embed the partially resolved ones as the parent singularities. A toric diagram corresponding to a partial resolution can of course be embedded in more than one parent singularity and according to the algorithm of [21], one can choose the minimal embedding. We will return to this issue in a while.

We now determine the fields to be resolved from the parent singularity in order to reach the given toric diagram, by appropriately choosing the values of ζ_i . The difficulty that arises here is that given the toric diagram of the singularity that we are interested in, there is no unique way of knowing exactly which fields (in the toric data of the parent singularity) have to be resolved in order to reach the toric singularity of interest. Given a toric diagram for the parent singularity, a particular resolution, along with the elimination of one or more nodes, might also require a subset of fields from some other nodes to be resolved. Hence, we have to carefully tune the FI parameters for a consistent blowup of the parent singularity to the one that we are interested in. We now illustrate this procedure by the example of the blowup of the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ singularity to the $\mathbf{C}^2/\mathbf{Z}_2 \times \mathbf{C}$ orbifold illustrated in fig. 1(c). This example is, in a sense, straightforward, because from the toric diagram of fig. 1(c), it is clear that this being the \mathbf{Z}_2 singularity, we can directly read off the matter content and the interactions from the standard results for D3-branes on the \mathbf{Z}_2 orbifold [8]. However, note that this is the simplest example where the inverse procedure of calculating the D-brane gauge theory has to be valid. The \mathbf{Z}_2 orbifold can be thought of as partial resolutions of the orbifolds $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$, or $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$, and the inverse toric procedure must give familiar results for both in agreement with [8],[23]. A lower dimensional toric orbifold can always be embedded in the toric diagram of one of higher dimension, and the inverse toric procedure must consistently reproduce the D-brane gauge theory on the former, starting from the latter data. As we will see, this is indeed the case.

We proceed by embedding the \mathbf{Z}_2 singularity in $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$. In order to determine the fields to be resolved, we perform Gaussian row reduction on

the the matrix Q_t to obtain

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & \zeta_1 + \zeta_2 + \zeta_3 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & \zeta_1 + \zeta_2 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & \zeta_1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & \zeta_1 + \zeta_3 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & \zeta_1 + \zeta_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \zeta_1 + \zeta_2 \end{pmatrix}$$

The above set of equations imply the constraints

$$\begin{aligned} -x_1 + x_6 - x_7 + x_9 &= (\zeta_1 + \zeta_2 + \zeta_3) \\ -x_2 - x_7 + 2x_9 &= (\zeta_1 + \zeta_2) \\ -x_3 + x_6 - x_7 + x_9 &= \zeta_1 \\ -x_4 + x_6 &= (\zeta_1 + \zeta_3) \\ -x_5 + 2x_6 - x_7 &= (\zeta_1 + \zeta_3) \\ -x_8 + x_9 &= (\zeta_1 + \zeta_2) \end{aligned} \tag{17}$$

Where, for convenience of notation, we have labelled $p_i^2 = x_i$. This set of equations is now solved in terms of the fields that we know would definitely get resolved. From the toric diagram, three such fields are p_7, p_8, p_9 . However, from the relation between x_8 and x_9 above, it suffices to solve the above set of equations in terms of x_7 and x_9 , say. This gives the solution set,

$$\begin{aligned} [x_1, \dots, x_9] &= [x_1, (2x_9 - x_7 - \zeta_1 - \zeta_2), (x_1 + \zeta_2 + \zeta_3), (x_1 + x_7 - x_9 + \zeta_2), \\ &\quad (2x_1 + x_7 - 2x_9 + \zeta_1 + 2\zeta_2 + \zeta_3)(x_1 + x_7 - x_9 + \zeta_1 + \zeta_2 + \zeta_3), \\ &\quad x_7, (x_9 - \zeta_1 - \zeta_2), x_9] \end{aligned} \tag{18}$$

Now, from the toric diagram of fig.1(c), we choose the fields $[p_1, p_2, p_3, p_5]$ to have zero vev i.e these fields continue to remain dynamical. Thus, in the above solution set, we impose $x_1, x_2, x_3, x_5 = 0$ to obtain

$$[x_1, \dots, x_9] = [0, 0, 0, (x_9 - \zeta_1), 0, (x_7 - x_9 + \zeta_1), x_7, (x_9 - \zeta_1 - \zeta_2), x_9] \tag{19}$$

We further set $x_9 = \zeta_1$ in order to make $x_4 = 0$. Hence, $x_6 = x_7 = \zeta_1 - \zeta_2$. Also, $x_8 = -\zeta_2$. Therefore, the fields to be resolved are $[p_6, p_7, p_8, p_9]$, all of which have positive vevs.

We now need to obtain the reduced charge matrix for this choice of variables to be resolved. The method of doing this essentially consists of performing row operations on the full charge matrix Q_t in eq. (14), such that the columns corresponding to the resolved fields $[p_6, p_7, p_8, p_9]$ are set to zero. The reduced charge matrix Q_t^{red} thus obtained must be in the nullspace of the reduced toric matrix \tilde{T}_{red} , that can be directly evaluated from the full toric matrix in eq. (15) by removing the columns 6, 7, 8, 9. Let us first discuss the general formalism for achieving this, which can then be easily applied to our present example. Consider a general parent theory that has a total charge matrix Q_t of dimensions $(a \times b)$, and suppose we wish to eliminate d fields from the parent theory in order to reach the theory of interest. This implies that we have to perform row operations on Q_t so that the given d columns will have zero entries and thus be eliminated. The latter can be achieved by constructing the nullspace of the transpose of a submatrix that is formed by precisely the d columns that need to be eliminated. Further, since the toric data \tilde{T} was initially in the nullspace of Q_t , removal of the d columns from Q_t to obtain Q_t^{red} would mean that the reduced toric data \tilde{T}_{red} with the same d columns removed would be in the nullspace of Q_t^{red} . Thus, the expression for Q_t^{red} is given by [21]

$$Q_t^{red} = [\text{NullSpace}(Q_d^t) \cdot Q_r] \quad (20)$$

where Q_d^t is the transpose of the matrix obtained from Q containing as its columns, the deleted columns of Q and Q_r is the remaining matrix.

In our example, we evaluate Q_{red} directly, by writing Q_d^t as the transpose of the matrix that contains columns 6, 7, 8, 9 of the charge matrix Q_t , and Q_r being the matrix containing the columns 1, 2, 3, 4, 5, 6, 10 of Q_t . This implies that the reduced charge matrix is

$$Q_t^{red} = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 0 & 1 & \zeta_2 + \zeta_3 \end{pmatrix} \quad (21)$$

From this, the reduced charge matrices corresponding to the F and D-terms are

$$Q_{red} = (1 \quad -1 \quad 1 \quad 0 \quad -1); \quad (VU)_{red} = (-2 \quad 1 \quad 0 \quad 0 \quad 1) \quad (22)$$

Denoting the kernel of the charge matrix by T_{red} , the transpose of its dual is

the reduced K^t matrix,

$$K_{red}^t = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Now, From eq. (6), the charge matrix for the initial fields X_i are given, via the charge matrix for the independent fields x_b by the relation

$$\Delta = V \cdot K^t \quad (23)$$

Hence, in this case, making use of the identity $U_{red} \cdot T_{red}^t = \text{Id}$, the matter content for the D-brane gauge theory on the resolved space is obtained as $\Delta_{red} = (VU)_{red}(T_{red}^t K_{red}^t)$. From which in this case we obtain the matrix

$$d_{red} = \begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix} \quad (24)$$

where we have included an extra row corresponding to the redundant $U(1)$. Hence, the gauge group is seen to be $U(1)^2$, with four bifundamentals, in agreement with the result of [23]. Also note that one adjoint field has appeared in our expression. This signals an ambiguity that exists in the toric procedure with regards to handling chargeless fields. Consider, for example, the superpotential interaction, which can be determined from the matrix K that encodes the F-flatness conditions. The number of columns in the matrix K_{red}^t is the number of fields that appear in the interactions and the terms in the superpotential are read off as linear relations between the columns of K_{red}^t . In this example, from the matrix K , we have the relation $X_2 X_5 = X_3 X_4$, and from the d matrix, we can see that each of these have charge zero. Hence, we try as an ansatz for the superpotential,

$$W = (X_1 + \phi) (X_2 X_5 - X_3 X_4) \quad (25)$$

where ϕ is a field uncharged for both the $U(1)$'s. This agrees with [23] after the identification $x_{13} \rightarrow X_2$; $Y_{34} \rightarrow X_5$; $X_{24} \rightarrow X_3$; $X_{13} \rightarrow X_4$; $Z_{41} \rightarrow X_1$; $Z_{23} \rightarrow -\phi$. As is clear, however, there is an ambiguity in writing the superpotential due to the presence of fields that are chargeless under both the $U(1)$'s [21]. Let us also mention that we can carry out the inverse procedure just outlined for the case of the alternative blowup to $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{C}$ shown in

figure 1(d). This gives exactly the same matter content and superpotential as the case just analysed, in the region of moduli space given by $\zeta_1 + \zeta_3 = 0$, $\zeta_2 \gg 0$.

- We now discuss the example of the blowup of the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ singularity to the conifold. Proceeding in the same way as above, we now find that in the region of the FI parameters determined by $\zeta_2 = 0$; $\zeta_{1,3} \gg 0$, the fields to be resolved are $[p_3, p_5, p_6, p_7, p_9]$, when the fields $[p_1, p_2, p_4, p_8]$ are chosen dynamical, i.e continue to have zero vev. Therefore, from eq. (20), we obtain the reduced charge matrix to be

$$Q_t^{red} = (-1 \ 1 \ 1 \ -1 \ \zeta_2) \quad (26)$$

Note that in this, case, there are no F-terms and we have $Q_{red} = \mathbf{0}_{1 \times 4}$, moreover, the kernel of this matrix is the 4×4 identity matrix, i.e $T_{red} = \text{Id}_{4 \times 4}$, which implies that (since the dual matrix of T_{red} is also the 4×4 identity matrix) $\Delta = Q_{red}$, with the charge matrix given in this case (by adding the extra column corresponding to the overall $U(1)$) as

$$d_{red} = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad (27)$$

In this case, the matrix Q_{red} being zero, there are no F-terms and the entire gauge theory information is contained in the D-term.

3 D-branes on $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ and its blowups

Let us now consider D3-branes on the orbifold $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$. In this case, the bosonic fields corresponding to the D3-brane coordinates tangential to the \mathbf{C}^3 are 6×6 matrices, and the components that survive the projection by the quotienting group constitute the $N = 1$ matter multiplet, given by the set

$$\begin{aligned} & (X_{12}, X_{23}, X_{34}, X_{45}, X_{56}, X_{61}) \\ & (Y_{13}, Y_{24}, Y_{35}, Y_{46}, Y_{51}, Y_{62}) \\ & (Z_{14}, Z_{25}, Z_{36}, Z_{41}, Z_{52}, Z_{63}) \end{aligned} \quad (28)$$

The F-flatness condition (5) in this case imply 18 constraints out of which 8 are seen to be independent. Further, there are 17 homogeneous coordinates

p_α , ($\alpha = 1 \cdots 17$), corresponding to the physical fields in terms of which the dual cone may be described.

The total charge matrix of the homogeneous coordinates p_α is defined by introducing a set of 9 \mathbf{C}^* actions that remove the redundancy in expressing the original independent fields in terms of the homogeneous coordinates and a further set of $U(1)^5$ charges expressing the D-term constraints in terms of the p_α . (The 6th $U(1)$ is redundant because of the relation $\sum_i \zeta_i = 0$). The total charge matrix can then be obtained by concatenating these two sets of charge matrices as in our earlier example, and is given (with the inclusion of a column specifying the 5 independent Fayet-Illiopoulos parameters) by [23]

$$Q_t = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_4 \\ -1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 \end{pmatrix}$$

The toric data is the co-kernel of the transpose of this charge matrix (when the FI parameters are set to zero) and is given by

$$\tilde{T} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & -2 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (29)$$

The invariant variables are defined by

$$\begin{aligned} X_{12}X_{23}X_{34}X_{45}X_{56}X_{61} = x &= p_1^4 p_2 p_3^6 p_4^2 p_5 p_6^3 p_7^2 p_8^4 p_9^3 p_{10}^4 p_{12} p_{13} p_{15}^2 p_{16} p_{17} \\ Y_{24}Y_{46}Y_{62} = y &= p_1 p_2 p_4^2 p_5 p_7^2 p_8 p_{10} p_{11}^3 p_{12} p_{13} p_{15}^2 p_{16} p_{17} \\ Z_{14}Z_{41} = z &= p_2 p_5 p_6 p_9 p_{12} p_{13} p_{14}^2 p_{16} p_{17} \\ X_{12}Y_{24}Z_{41} = w &= p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9 p_{10} p_{11} p_{12} p_{13} \\ &\quad p_{14} p_{15} p_{16} p_{17} \end{aligned} \quad (30)$$

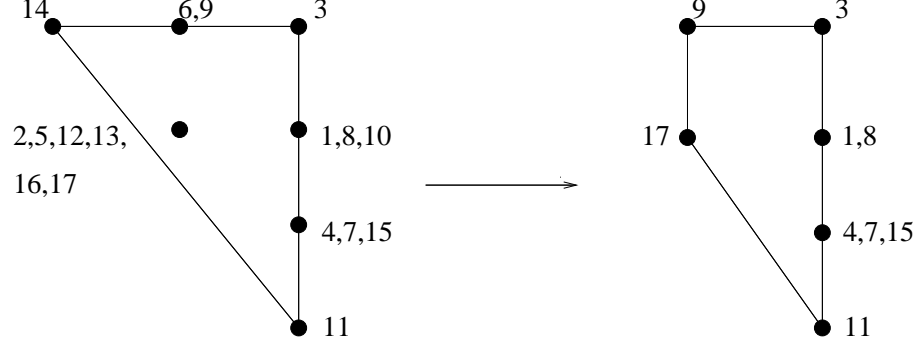


Figure 2: Toric diagram showing a resolution of the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ orbifold along with the choice of fields to be resolved.

From which it can be seen that

$$xy^2z^3 = w^6 \quad (31)$$

Let us now discuss some examples of blowups of this space that will illustrate the procedure outlined in section 2. We will use the notation $x_i = p_i^2$ in what follows.

- Our first example is illustrated in figure 2. From the diagram, we see that the field p_{14} will definitely be resolved. Hence, after performing Gaussian row reduction on the charge matrix Q_t , which gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & -\zeta_1 - 2\zeta_2 - \zeta_4 - 2\zeta_5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\zeta_1 - \zeta_2 - \zeta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -3 & 0 & 0 & -\zeta_1 - 2\zeta_2 - \zeta_4 - 2\zeta_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\zeta_2 - \zeta_5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\zeta_2 - \zeta_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -\zeta_1 - \zeta_2 - \zeta_3 - \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\zeta_1 - \zeta_2 - \zeta_4 - \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -\zeta_2 - \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -\zeta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -\zeta_1 - \zeta_2 - \zeta_4 - \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & \zeta_4 + \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & \zeta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & -\zeta_3 + \zeta_4 + \zeta_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -\zeta_3 \end{pmatrix} \quad (32)$$

we use this matrix to solve for all the fields in terms of x_{14} , the solution set being given by

$$\begin{aligned}
[x_1 \cdots x_{17}] = & [x_1, x_2, (2x_1 - 2x_2 + x_{14} - \zeta_1 - \zeta_3 + \zeta_5), \\
& (2x_2 - x_{14} + 2\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), (x_2 + \zeta_1), \\
& (x_1 - x_2 + x_{14} - \zeta_1 - \zeta_3), (2x_2 - x_{14} + \zeta_1 + \zeta_2 + \zeta_3), \\
& (x_1 + \zeta_1 + \zeta_2 + \zeta_4 + \zeta_5), (x_1 - x_2 + x_{14} + \zeta_5), \\
& (x_1 + \zeta_2 + \zeta_5), (-x_1 + 4x_2 - 2x_{14} + 3\zeta_1 + 2\zeta_2 + 2\zeta_3 + \zeta_4), \\
& (x_2 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5), (x_2 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), \\
& x_{14}, (2x_2 - x_{14} + 2\zeta_1 + 2\zeta_2 + \zeta_3 + \zeta_4 + \zeta_5), \\
& (x_2 + \zeta_1 + \zeta_2), (x_2 + \zeta_1 + \zeta_2 + \zeta_3)] \quad (33)
\end{aligned}$$

Now, we choose the fields $[p_1, p_3, p_4, p_{11}]$ as the ones that continue to have zero vev, and in eq. (33) set the values of these to zero. We thus obtain conditions on the FI parameters, $\zeta_1 + \zeta_2 + \zeta_4 = 0$, $\zeta_2 = 2\zeta_5$. In the new solution set obtained with these conditions, we choose $\zeta_5 = 0$ which makes $x_8 = 0$. This also implies that $\zeta_2 = 0$ (since $\zeta_2 = 2\zeta_5$). We also set $x_2 = x_{14} = -\zeta_3 - \zeta_1$ to obtain $x_7, x_9, x_{15}, x_{17} = 0$. Our choice of $\zeta_2 = \zeta_5 = 0$ implies that $x_{10} = 0$. This gives the solution set

$$\begin{aligned}
[x_1 \cdots x_{17}] = & [0, (-\zeta_3 - \zeta_1), 0, 0, -\zeta_3, (-\zeta_1 - \zeta_3), \\
& 0, 0, 0, 0, 0, -\zeta_1, -\zeta_1, (-\zeta_3 - \zeta_1), 0, -\zeta_3, 0] \quad (34)
\end{aligned}$$

Thus the fields to be resolved are $[p_2, p_5, p_6, p_{12}, p_{13}, p_{14}, p_{16}]$. Hence, from (20), we obtain the reduced charge matrix,

$$Q_t^{red} = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & \zeta_2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 + \zeta_4 \\ -1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & -1 & \zeta_5 \end{pmatrix}$$

In the usual way, we calculate the kernel of the charge matrix T_{red} , and

the transpose of its dual matrix is

$$K_{red}^t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

from the matrices T_{red} , K_{red}^t and $(VU)_{red}$, we can read off the matter content and gauge group of the D-brane gauge theory from the matrix

$$d_{red} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

This is an $U(1)^4$ gauge theory with 8 matter fields, in agreement with [23],[24]. To calculate the superpotential, we first note that the relations between the columns of the matrix K^t , which can be written as $X_3X_7 = X_5X_6 = X_4X_8$. All these combinations are seen to be chargeless, as is the combination X_1X_2 . Hence, our ansatz for the superpotential in this case is

$$W = X_1X_2(X_3X_7 - X_5X_6) + \phi_1(X_3X_7 - X_4X_8) + \phi_2(X_5X_6 - X_4X_8) \quad (35)$$

which agrees with that calculated in [23] after the identification with their notation, $Z_{25} \rightarrow X_1, Z_{52} \rightarrow X_2; X_{23} \rightarrow X_3; Y_{62} \rightarrow X_7; Y_{51} \rightarrow X_5; X_{45} \rightarrow X_6; Y \rightarrow X_4; X_{61} \rightarrow X_8; Z_{36} \rightarrow -\phi_1; Z_{14} \rightarrow \phi_2$.

In terms of the original variables of (28), this blowup corresponds to giving a vev to the fields Z_{63}, Z_{41} . The invariant variables are in this case defined by

$$\begin{aligned} x' &= p_1^2 p_3^3 p_4 p_7 p_8^2 p_9 p_{10}^2 p_{15}; & x &= x'^2 z \\ y &= p_1 p_4^2 p_7^2 p_8 p_{10} p_{11}^3 p_{15}^2 p_{17} \\ z &= p_9 p_{17} \\ w' &= p_1 p_3 p_4 p_7 p_8 p_{10} p_{11} p_{15}; & w &= w' z \end{aligned} \quad (36)$$

In terms of these variables, the blown up space is given by $x'y = zw'^3$.

• We now consider the second example which is depicted in figure 3(a). In this case, we start from the matrix (32), and in the solution set of eq. (33), keeping in mind the toric diagram of the blowup, we select the set of fields $[x_7, x_9]$ that remain dynamical, with zero vev. This gives the solution $x_1 = \zeta_1 + \zeta_4$. From the resulting solution set, we now select $x_{17} = 0$, in order to obtain $x_2 = x_{14} = -\zeta_1 - \zeta_2 - \zeta_3$ and after imposing this condition, we make the choice $\zeta_1 + \zeta_4 + \zeta_5 = 0$ in order for p_9 to have zero vev. The final solution is given by

$$\begin{aligned} [x_1 \cdots x_{17}] = & [(\zeta_1 + \zeta_4), (-\zeta_1 - \zeta_2 - \zeta_3), (\zeta_1 + \zeta_2 + \zeta_4), (\zeta_1 + \zeta_4), \\ & (\zeta_2 - \zeta_3), (\zeta_4 - \zeta_3), 0, (\zeta_1 + \zeta_2 + \zeta_4), 0, \zeta_2, 0, -\zeta_1, \zeta_4, \\ & (-\zeta_1 - \zeta_2 - \zeta_3), \zeta_2, -\zeta_3, 0] \end{aligned} \quad (37)$$

Hence, the fields to be resolved are $[p_1, p_2, p_3, p_4, p_5, p_6, p_8, p_{10}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}]$ and the dynamical fields are $[p_7, p_9, p_{11}, p_{17}]$. We can thus directly evaluate the reduced charge matrix from (20) which is given by

$$Q_t^{red} = (-1 \ 1 \ 1 \ -1 \ \zeta_1 + \zeta_4 + \zeta_5) \quad (38)$$

Defining the invariant variables from eq. (28) in this case as

$$\begin{aligned} x' &= p_7 p_9; & x &= x'^2 z \\ y' &= p_{11} p_{17}; & y &= y' w'^2 \\ z &= p_9 p_{17} \\ w' &= p_7 p_{11}; & w &= w' z \end{aligned} \quad (39)$$

this blowup space can be seen to be the conifold $x'y' = zw'$ by using the redefined variables in (31). The gauge theory living on the D-brane is of course the same as in eq. (27). In terms of the original variables in eq. (28), this blowup can be shown to correspond to giving vevs to the fields $X_{45}, X_{23}, Z_{63}, Z_{41}$. Since there are no F-terms in this case, all the information about the gauge theory is contained in the D-term constraint.

• Let us now consider the toric diagram given by figure 3(b). From the toric diagram of fig. 3(b), we choose the fields $p_1, p_2, p_3, p_4, p_{11}$ to have zero vev. These conditions imply that $\zeta_1 + \zeta_4 = 0$ and also $\zeta_2 + \zeta_5 = 0$. Substituting

this in the solution set (33), we obtain

$$\begin{aligned} [x_1 \cdots x_{17}] &= [0, 0, 0, 0, \zeta_1, \zeta_2, 0, 0, (\zeta_1 + \zeta_3), 0, 0, \zeta_3, (\zeta_2 + \zeta_3), \\ &\quad (\zeta_1 + \zeta_3 - \zeta_5), 0, (\zeta_1 + \zeta_2), (\zeta_1 + \zeta_2 + \zeta_3)] \end{aligned} \quad (40)$$

Now, we have a choice. We put $\zeta_1 = 0$ and obtain $x_5 = 0$ which also implies that $\zeta_4 = 0$. Hence, from the final solution set, we see that the fields that continue to have zero vev and are dynamical are, in this case, given by the set $[p_1, p_2, p_3, p_4, p_5, p_7, p_8, p_{10}, p_{11}, p_{15}]$ and the fields that pick up non-zero vev and hence are resolved are $[p_6, p_9, p_{12}, p_{13}, p_{14}, p_{16}, p_{17}]$. Hence, according to (20), we obtain the reduced charge matrix as

$$Q_t^{red} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \zeta_2 + \zeta_5 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & \zeta_4 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \end{pmatrix}$$

and the dual of the kernel of the charge matrix Q_{red} is

$$K_{red}^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As before, in order to calculate the matter content, we use the matrix $T_{red}^t \cdot K_{red}^t$, from which the matrix Δ , which specifies the matter content and gauge group of the D-brane gauge theory is calculated to be

$$d_{red} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Hence, the gauge group is $U(1)^4$ and there are 8 matter fields. Using the relations $X_1 X_4 = X_5 X_6 = X_7 X_8$ with total charge zero and the chargeless combination $X_2 X_3$, we write the superpotential in this case as

$$W = X_2 X_3 (X_1 X_4 - X_5 X_6) + \phi_1 (X_1 X_4 - X_7 X_8) + \phi_2 (X_5 X_6 - X_7 X_8) \quad (41)$$

where the fields ϕ_1, ϕ_2 are possible adjoints. The invariant variables (30) are defined by

$$\begin{aligned} x' &= p_1^2 p_3^3 p_4 p_7 p_8^2 p_{10}^2 p_{15} \\ y' &= p_1 p_4^2 p_7^2 p_8 p_{10} p_{11}^3 p_{15}^2 \\ z &= p_2 p_5 \\ w' &= p_1 p_3 p_4 p_7 p_8 p_{10} p_{11} p_{15} \end{aligned} \quad (42)$$

with the relations $x = x'^2 z$; $y = y' z$; $w = w' z$. The remaining singularity is seen from eq. (31) to be the equation $x' y' = w'^3$ in \mathbf{C}^3 , which is recognised as corresponding to blowing up the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ orbifold to the singularity $\mathbf{C}^2/\mathbf{Z}_3 \times \mathbf{C}$. Let us also mention that in terms of the original variables in eq. (28), this blowup corresponds to giving vevs to the fields $Y_{13}, Z_{14}, Z_{52}, Z_{63}$.

- We now come to the example shown in figure 3(c). In this case, from the Gaussian row-reduced matrix of eq. (33), we choose the set of fields $[p_{10}, p_7, p_{11}]$ to have zero vev. This implies that $\zeta_1 + \zeta_2 + \zeta_4 + \zeta_5 = 0$ and also that $x_1 = \zeta_1 + \zeta_4$. Substituting this in the solution set implies that in addition $x_{15} = 0$, and further setting $x_2 = -\zeta_3 + \zeta_4 + \zeta_5$, so that x_{17} is zero, we obtain the full solution set in this case as

$$\begin{aligned} [x_1 \cdots x_{17}] &= [(-\zeta_2 - \zeta_5), (-\zeta_3 + \zeta_4 + \zeta_5), (-\zeta_2 - \zeta_5), (-\zeta_2 - \zeta_5), \\ &\quad (-\zeta_2 - \zeta_3), (\zeta_4 - \zeta_5), 0, (-\zeta_2 - \zeta_5), -\zeta_2, 0, 0, (\zeta_4 + \zeta_5), \zeta_4, \\ &\quad (-\zeta_3 + \zeta_4 + \zeta_5), 0, -\zeta_3, 0] \end{aligned} \quad (43)$$

Hence, the fields that are dynamical and have zero vev is given by the set $[p_7, p_{10}, p_{11}, p_{15}, p_{17}]$ and the set to be resolved is given by $[p_1, p_2, p_3, p_4, p_5, p_6, p_8, p_9, p_{12}, p_{13}, p_{14}, p_{16}]$. Using this, the reduced charge matrix is calculated from (20) to be

$$Q_t^{red} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & \zeta_1 + \zeta_2 + \zeta_4 + \zeta_5 \end{pmatrix}$$

From which the F and the D-terms as

$$Q_{red} = (1 \ -1 \ -1 \ 1 \ 0) \quad (VU)_{red} = (0 \ -1 \ -1 \ 2 \ 0) \quad (44)$$

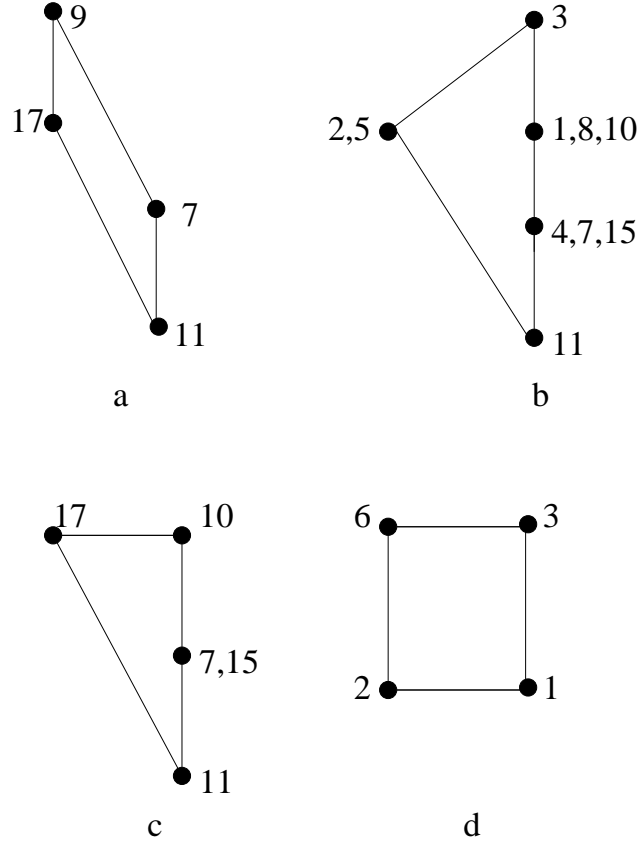


Figure 3: Toric diagram showing some resolutions of the $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ orbifold that we have considered. We have marked the various fields that are chosen to remain dynamical on resolving the parent singularity to these cases.

The dual of the kernel of Q_{red} is given by

$$K_{red}^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and the matter content and gauge group of the D-brane gauge theory is obtained as

$$d_{red} = \begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix}$$

The superpotential is written down, after noting that the field X_1 is chargeless, and the relations $X_2X_5 = X_3X_4$ from the matrix K^t . It is given by

$$W = (X_1 + \phi)(X_2X_5 - X_3X_4) \quad (45)$$

where ϕ is a possible adjoint. The matter content and the superpotential are exactly the same as in (24) and (25). This is of course as expected because both describe the world volume theory of a D3-brane on the $\mathbf{C}^2/\mathbf{Z}_2$ singularity. This blowup corresponds to giving vevs to the fields X_{12}, X_{23}, X_{45} . The invariant variables (30) are in this case

$$\begin{aligned} x' &= p_7 p_{10}^2 p_{15} \\ y' &= p_7 p_{11}^2 p_{15} \\ z &= p_{17} \\ w' &= p_7 p_{10} p_{11} p_{15} \end{aligned} \quad (46)$$

where $x = x'^2 z; y = y' w' z; w = w' z$. The blown up space is, from eq. (31), given by the equation $x' y' = w'^2$ in \mathbf{C}^3 , which describes the blowup to $\mathbf{C}^2/\mathbf{Z}_3 \times \mathbf{C}$. Let us pause to comment here that the toric diagram of figure 3(c) can be embedded both into that of $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_3$ and $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$; in fact, this diagram is identical to that in figure 1(d). The fact that either embedding gives correct results for the D-brane gauge theory data shows that the inverse toric procedure is indeed consistent.

- Finally, let us discuss the toric diagram of figure 3(d). which is the conifold obtained by partially resolving the $\mathbf{C}^2/\mathbf{Z}_2 \times \mathbf{Z}_3$ singularity. Following the

same procedure as outlined in the previous examples, we start from the solution set of (33) and select the fields $[p_1, p_3, p_6]$ which we set to zero. From this, we obtain the solution $x_2 = \zeta_5$ and $x_{14} = (\zeta_1 + \zeta_2 + \zeta_5)$. Substituting this in the solution set and further setting $\zeta_5 = 0$ so that $x_2 = 0$, we finally obtain the full solution set to be

$$\begin{aligned} [x_1 \cdots x_{17}] = & [0, 0, 0, (\zeta_1 + \zeta_2 + \zeta_4), \zeta_1, 0, \zeta_2, (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), (\zeta_1 + \zeta_3), \\ & \zeta_2, (\zeta_1 + 2\zeta_2 + \zeta_4)(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), \\ & (\zeta_1 + \zeta_3), (\zeta_1 + 2\zeta_2 + \zeta_4), (\zeta_1 + \zeta_2), (\zeta_1 + \zeta_2 + \zeta_3)] \end{aligned} \quad (47)$$

Hence, the fields that are dynamical are given by the set $[p_1, p_2, p_3, p_6]$ while the rest acquire non-zero vev and hence are resolved, and in this case the reduced charge matrix is obtained from (20) to be

$$Q = \begin{pmatrix} -1 & 1 & 1 & -1 & \zeta_5 \end{pmatrix} \quad (48)$$

This blowup corresponds to the conifold. To see this, we define the invariant variables (30) in this case by

$$x' = p_1 p_3 = \frac{w}{z}; \quad y = p_1 p_2; \quad z = p_2 p_6; \quad w' = p_3 p_6 \quad (49)$$

In this case, from (31), we can see that the remaining singularity is given by the conifold $x'y = zw'$. Let us also mention that in terms of the original variables in eq. (28), this blowup corresponds to giving vevs to Y_{13}, Y_{24}, Z_{36} .

4 D-branes on $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and its blowups

We now consider D-branes on the orbifold $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and some of its blowups [26]. We will consider a D1-brane at this singularity, and the world volume theory is an $N = (0, 2)$ SYM theory in two dimensions. The method of construction of the moduli space of the D-1 brane is similar to the method outlined in sections 2 and 3. The unbroken gauge symmetry in this case is (apart from a redundant $U(1)$), $U(1)^7$. The matter multiplet, after imposing the appropriate projections consist of 32 surviving fields,

$$\begin{aligned} & (X_{18}, X_{27}, X_{36}, X_{45}, X_{54}, X_{63}, X_{72}, X_{81}), (Y_{15}, Y_{26}, Y_{37}, Y_{48}, Y_{51}, Y_{62}, Y_{73}, Y_{84}) \\ & (Z_{13}, Z_{24}, Z_{31}, Z_{42}, Z_{57}, Z_{68}, Z_{75}, Z_{86}), (W_{12}, W_{21}, W_{34}, W_{43}, W_{56}, W_{65}, W_{78}, W_{87}) \end{aligned} \quad (50)$$

The F-term constraints of eq. (5), in this case imply that out of the 32 fields, 11 are independent, and the 32 initial fields can be solved in terms of these 11, and the solutions can be denoted as vectors in the lattice \mathbf{Z}^{11} . The dual cone in this case has 34 homogeneous coordinates (matter fields) p_α , and the redundancy in the definition of the initial independent coordinates in terms of these can be eliminated by introducing a set of 23 \mathbf{C}^* actions. The charge matrix Q for the \mathbf{C}^* actions is given in [26]. For the matrix denoting the original $U(1)$ charges expressed in terms of the homogeneous coordinates p_α , we use the Moore-Penrose inverse of the matrix defining the dual cone, and concatenate these two matrices to obtain the total charge matrix given in the appendix. The toric diagram is given by the columns of the matrix

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} \\ 1 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \dots\dots \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dots \begin{pmatrix} p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The distinct columns in the toric data are shown in fig. 4. The labelling of the vectors and the corresponding p_α are as follows

$$\begin{array}{lll} a & (2 \ 0 \ 0 \ 1) & [p_6] \\ b & (1 \ 0 \ 0 \ 1) & [p_7, p_{13}] \\ c & (1 \ 0 \ 1 \ 0) & [p_5, p_{10}] \\ d & (1 \ 1 \ 1 \ 0) & [p_1, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}, p_{22}, p_{23}, \\ & & p_{24}, p_{25}, p_{26}, p_{27}, p_{29}, p_{30}, p_{31}, p_{32}, p_{33}, p_{34}] \\ e & (1 \ 1 \ 0 \ 0) & [p_3, p_{12}] \\ f & (0 \ 0 \ 0 \ 1) & [p_{20}] \\ g & (0 \ 0 \ 1 \ 0) & [p_8, p_{21}] \\ h & (0 \ 0 \ 2 \ -1) & [p_9] \end{array}$$

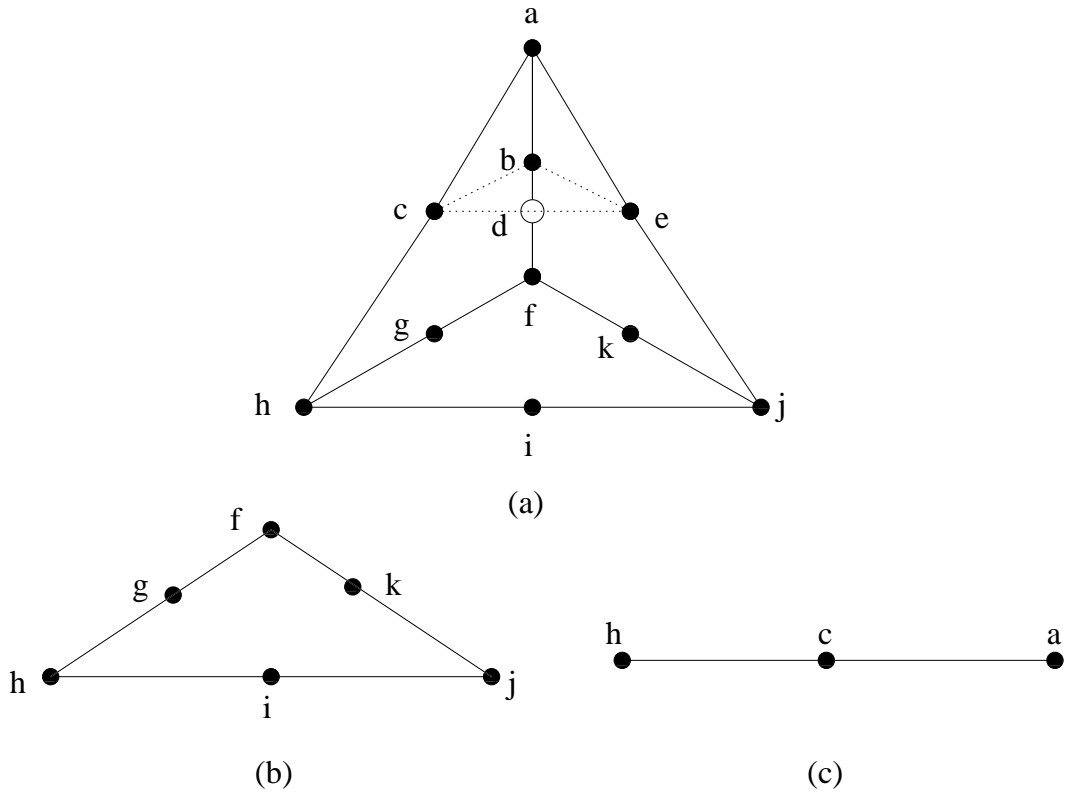


Figure 4: Toric data for the orbifold $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and two of its blowups to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ singularity and the \mathbf{Z}_2 singularity.

$$\begin{array}{lll}
i & (0 \ 1 \ 1 \ -1) & [p_4, p_{19}] \\
j & (0 \ 2 \ 0 \ -1) & [p_{11}] \\
k & (0 \ 1 \ 0 \ 0) & [p_{11}]
\end{array} \tag{51}$$

The invariant variables, in this case denoted by $x = (X_{18}X_{81})$; $y = (Y_{15}Y_{51})$; $z = (Z_{13}Z_{31})$; $w = (W_{12}W_{21})$; $v = (X_{18}Y_{84}Z_{42}W_{21})$, are

$$\begin{aligned}
x &= p_1 p_2 p_7 p_8 p_{13} p_{14} p_{15} p_{16} p_{17} p_{18} p_{20}^2 p_{21} p_{22} p_{23} p_{24} \\
&\quad p_{25} p_{26} p_{27} p_{28} p_{29} p_{30} p_{31} p_{32} p_{33} p_{34} \\
y &= p_1 p_2 p_3 p_4 p_{11}^2 p_{12} p_{14} p_{15} p_{16} p_{17} p_{18} p_{19} p_{22} p_{23} p_{24} \\
&\quad p_{25} p_{26} p_{27} p_{28} p_{29} p_{30} p_{31} p_{32} p_{33} p_{34} \\
z &= p_1 p_4 p_5 p_8 p_9^2 p_{10} p_{14} p_{15} p_{16} p_{17} p_{18} p_{19} p_{21} p_{22} p_{23} p_{24} \\
&\quad p_{25} p_{26} p_{27} p_{29} p_{30} p_{31} p_{32} p_{33} p_{34} \\
w &= p_1 p_3 p_5 p_6^2 p_7 p_{10} p_{12} p_{13} p_{14} p_{15} p_{16} p_{17} p_{18} p_{22} p_{23} p_{24} \\
&\quad p_{25} p_{26} p_{27} p_{29} p_{30} p_{31} p_{32} p_{33} p_{34} \\
v &= p_1^2 p_2 p_3 p_4 p_5 p_6 p_7 p_8 p_9 p_{10} p_{11} p_{12} p_{13} p_{14}^2 p_{15}^2 p_{16}^2 p_{17}^2 \\
&\quad p_{18}^2 p_{19} p_{20} p_{21} p_{22}^2 p_{23}^2 p_{24}^2 p_{25}^2 p_{26}^2 p_{27}^2 p_{28}^2 p_{29}^2 p_{30}^2 p_{31}^2 p_{32}^2 p_{33}^2 p_{34}^2
\end{aligned} \tag{52}$$

In terms of these variables, the space is defined by the surface $xyzw = v^2$ in \mathbf{C}^5 . Two of its resolutions (corresponding to cases (Vb and VIb of [26]) are shown in figure 4.

This singularity has to be analysed in the lines of [22] in order to determine whether its partial resolutions are realised in the moduli space of the D-brane world volume gauge theory. We leave this issue for future work, and for the moment analyse the blowups of this singularity into lower dimensional orbifold singularities, using the inverse toric procedure.

- Let us consider the blowup illustrated in fig. 4(b). Performing Gaussian row reduction on the total charge matrix, we find that for the range of the FI parameters $\zeta_1 + \zeta_2 = 0$; $\zeta_3 + \zeta_4 = 0$; $\zeta_5 + \zeta_6 = 0$, an appropriate initial choice determines the fields that retain zero vev as the set $[p_2, p_4, p_8, p_9, p_{11}, p_{19}, p_{20}, p_{21}, p_{28}]$, while all others have non-zero vev and are

hence resolved. The reduced charge matrix is

$$Q_t^{red} = \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 5 & 5 & -5 & 0 & -4 & -1 & 0 & 1 & -1 & \zeta_5 + \zeta_6 \\ -5 & 5 & 5 & -4 & 0 & -1 & 0 & -1 & 1 & \zeta_3 + \zeta_4 \\ -1 & -5 & -5 & 4 & 0 & 1 & 0 & 1 & 5 & \zeta_1 + \zeta_2 \end{pmatrix}$$

From the kernel of the charge matrix Q_{red} , the transpose of its dual is obtained as

$$K^t = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From these matrices, we can extract the gauge theory data which is now given by the matrix

$$d_{red} = \begin{pmatrix} 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The gauge group is as expected, $U(1)^4$ with 12 matter fields, and the superpotential is given by

$$\begin{aligned} W &= X_1 X_5 X_{12} - X_3 X_4 X_{12} + X_2 X_8 X_9 - X_2 X_7 X_{11} + X_3 X_6 X_{11} \\ &\quad - X_5 X_6 X_9 + X_{10} X_4 X_7 - X_1 X_8 X_{10} \end{aligned} \quad (53)$$

From the definition of the invariant variables in eq. (52), it can be seen that this space is $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2$ defined by the equation $xyz = v^2$ in \mathbf{C}^4 . In terms of the original fields, it corresponds to giving vevs to the fields W_{12} and W_{21} .

• Finally, we come to the example of fig. 4(c). In this case, in the region of moduli space given by $\zeta_2 + \zeta_3 + \zeta_6 + \zeta_7 = 0$, we find that the dynamical fields are given by the set $[p_5, p_6, p_9, p_{10}]$, while the rest acquire non-zero vev and are resolved. The reduced charge matrix is

$$Q_t^{red} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 5 & -2 & -2 & -1 & \zeta_2 + \zeta_3 + \zeta_6 + \zeta_7 \end{pmatrix}$$

The matrix K^t is in this case given by

$$K^t = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Which specifies the matter content as a $U(1)^2$ gauge theory with the charge matrix given, as expected, by

$$d_{red} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

With a superpotential $W = \phi(X_1X_4 - X_2X_3)$ with ϕ being a possible chargeless field. In terms of the original fields in the theory, this resolution corresponds to giving vevs to $X_{18}, X_{81}, Y_{15}, Y_{51}$, and the resulting space is the $\mathbf{C}^2/\mathbf{Z}_2$ orbifold $zw = v^2$ that we have discussed earlier.

5 Discussions

In this paper, we have critically examined the inverse procedure of obtaining the world volume gauge theory data of D-branes probing certain toric singularities and their blowups from the geometric data of the resolution. We have shown that the algorithm of [21] gives consistent results, for a given singularity that can be reached by partial resolutions of different parent theories. We have explicitly checked several partial resolutions of orbifolds of the form \mathbf{C}^3/Γ and \mathbf{C}^4/Γ and found the results to be in agreement with field theory calculations. However, as pointed out in [21], in the presence of chargeless fields in the matter multiplet, the procedure cannot be used to specify the superpotential interaction uniquely, because of the inherent problem in the handling of such fields by toric methods.

We have treated the simplest examples of the resolution of the $\mathbf{C}^4/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ singularity in this paper. It would be interesting to investigate the various partial resolutions of the moduli space of D-branes on this singularity the lines of [22] and determine the gauge theory data thereof.

Finally, as noted in [21], the ambiguities that exist in the inverse algorithm seem to imply that in some cases, different gauge theories, in the infra red limit flow to theories with identical moduli space. We have not dealt with this aspect in the present paper, and it would be an interesting direction for future work.

Acknowledgements

I would like to thank I. Biswas, S. Bhattacharjee, P. Chatterjee and D. Suryaramana for helpful discussions.

Appendix

Charge matrix for the partial resolutions of $\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$:

[illegible]

	P18	P19	P20	P21	P22	P23	P24	P25	P26	P27	P28	P29	P30	P31	P32	P33	P34	P35
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
	1	2	0	2	2	-3	-1	1	3	-2	2	-1	1	3	3	-1	5	ζ_1
-3	2	0	2	-2	1	3	-3	-1	2	2	3	5	-1	-1	3	1	ζ_2	
-3	-2	0	-2	-2	1	-5	5	-1	2	2	3	-3	-1	-1	3	1	ζ_3	
1	-2	0	-2	2	5	-1	1	-5	-2	2	-1	1	3	3	-1	-3	ζ_4	
-3	-2	0	2	-2	1	3	-3	-1	2	-2	3	-3	-1	-1	-5	1	ζ_5	
1	-2	0	2	2	-3	-1	1	3	-2	-2	-1	1	3	-5	-1	-3	ζ_6	
1	2	0	-2	2	-3	-1	1	3	-1	-1	-1	1	-5	3	-1	-3	ζ_7	

References

- [1] M. R. Douglas, D. Kabat, P. Pouliot, S. Shenker, “D-branes and Short Distances in String Theory,” Nucl. Phys. **B485** (1997) 85, [hep-th/9608024](#).
- [2] B. Greene, Y. Kanter, “Small Volumes in Compactified String Theory,” Nucl. Phys. **B497** (1997) 127, [hep-th/9612181](#).
- [3] P. Aspinwall, B. Greene, D. Morrison, “Calabi-Yau Moduli Space, Mirror Manifolds and Space-time Topology Change in String Theory,” Nucl. Phys. **B416** (1994) 78, [hep-th/9309097](#).
- [4] E. Witten, “Phases of $N = 2$ Theories in Two Dimensions,” Nucl. Phys. **B403** (1993) 159, [hep-th/9301042](#).
- [5] M. Douglas, B. Greene, D. Morrison, “Orbifold Resolution by D-branes,” Nucl. Phys. **B506** (1997) 84, [hep-th/9704151](#).
- [6] E. Witten, “Phase Transitions in M Theory and F Theory,” Nucl. Phys. **B471** (1996) 195, [hep-th/9603150](#).
- [7] T. Muto, “D-branes on Orbifolds and Topology Change,” Nucl. Phys. **B521** (1998) 183, [hep-th/9711090](#).
- [8] M. Douglas, G. Moore, “D-branes, Quivers, and ALE Instantons,” [hep-th/9603167](#).
- [9] C. Johnson, R. Myers, “Aspects of Type IIB Theory on ALE Spaces,” Phys. Rev. **D55** (1997) 6382, [hep-th/9610140](#).
- [10] M. Douglas, B. Greene, “Metrics on D-brane Orbifolds,” Adv. Theor. Math. Phys. **1** (1998) 184, [hep-th/9707214](#).
- [11] M. Douglas, A. Kato, H. Ooguri, Adv. Theor. Math. Phys. **1** (1998) 237, [hep-th/9708012](#).
- [12] T. Muto, “Brane Configurations for Three-Dimensional Nonabelian Orbifolds,” [hep-th/9905230](#).
- [13] B. Feng, A. Hanany, Y. He, “Z-D-brane Box Models and Nonchiral Dihedral Quivers,” [hep-th/9909125](#).

- [14] K. Oh, R. Tatar, “Branes at Orbifolded Conifold Singularities and Supersymmetric Gauge Field Theories,” JHEP **9910i** (1999) 0319, [hep-th/9906012](#)
- [15] M. Cvetič, L. Everett, P. Langacker, J. Wang, “Blowing up the Four-Dimensional $Z(3)$ Orientifold,” JHEP **9904** (1999) 020, [hep-th/9903051](#)
- [16] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.
- [17] D. Cox, [alg-geom/9606016](#).
- [18] E. Gimon, J. Polchinski, “Consistency Conditions for Orientifolds and D Manifolds,” Phys. Rev. **D54** (1996) 1667, [hep-th/9601038](#).
- [19] B. Greene, “D-brane Topology Changing Transitions,” Nucl.Phys. **B525** (1998) 284, [hep-th/9711124](#).
- [20] S. Mukhopadhyay, K. Ray, “Conifolds from D-branes,” Phys. Lett. **B423** (1998) 247, [hep-th/9711131](#).
- [21] B. Feng, A. Hanany, Y. He, [hep-th/0003085](#).
- [22] C. Beasley, B. Greene, C. Lazaroiu, M. Plesser, “D3-branes on Partial Resolutions of Abelian Quotient Singularities of Calabi-Yau Threefolds,” Nucl. Phys. **B553** (1999) 711, [hep-th/9907186](#).
- [23] J. Park, R. Rabadan, A. M. Uranga, [hep-th/9907086](#), “Orientifolding the Conifold,” Nucl.Phys. **B570** (2000) 38, [hep-th/9907086](#).
- [24] A. Uranga, “Brane Configurations for Branes at Conifolds,” JHEP **9901** (1999) 022, [hep-th/9811004](#)
- [25] D. Morrison, R. Plesser, “Nonspherical Horizons 1,” Adv. Theor. Math. Phys. **3** (1999) 1, [hep-th/9810201](#).
- [26] A. Ahn, H. Kim, “Branes at C^4/Γ Singularity from Toric Geometry,” JHEP **9904** (1999) 012, [hep-th/9903181](#).
- [27] K. Mohri, “D-branes and Quotient Singularities of Calabi-Yau Four-folds,” Nucl. Phys. **B521** (1998) 161, [hep-th/9707012](#)